International Journal of Theoretical Physics, Vol. 6, No. 4 (1972), pp. 259–274

The Relativistic Riemann Invariants and the Propagation of Initial Data Surfaces

ANGELO JAMES SKALAFURIS

Bartol Research Foundation of The Franklin Institute, Swarthmore, Pennsylvania 19081

Received: 11 *October* 1971

Abstract

A method is described by which the relativistic Riemann invariants can be found for a fluid with an arbitrary equation of state, undergoing dissipation and moving in a general metric. Specific formulae are derived for a spherically symmetric system. Limiting cases defined by relativistic and non-relativistic gases, both warm, cold, fast and slow are examined. We prove that the invariants do exist, and a necessary and sufficient condition for their determination is the solution of a differential equation with the structure of an exterior one form of two components. The common parameter of these components is the characteristic space–time direction which is also derived in the process of determining the invariants. The characteristic surfaces, being the surfaces over which initial data is carried, all coalesce to the forward light cone in the extreme relativistic limit. Relativistic fluids emanating from receding sources appear to increase their internal kinetic energy as they decelerate.

A non-linear distance–velocity relation for these waves is evident in the differential equations which are found. Their full meaning remains to be explored.

1. *Introduction*

A set of partial differential equations may be rotated into a diagonal form such that they become an ordinary set. This transformation may be called the characteristic transformation of the set since the resulting solutions of the ordinary equations are known as the characteristic solutions, or characteristic curves. These curves determine the compatibility of an imposed set of boundary conditions with the uniqueness of the solution in the domain of interest. For example, the classical Poisson partial differential equation, by nature of its characteristic curves, requires only that the function or its gradient be specified at the boundary in order for the field to be determined in the domain of interest. This type of characteristic is called a Dirichlet or Neumann problem, depending upon the choice of boundary data. Once this choice is made, then the symmetries of the boundary are projected into the region of interest and may be found by geometrical techniques, such as the method of conformal mapping. On the other hand, the classical wave equation requires the specification of both the function and its derivative at the boundary, consistent with Newtonian dynamics, and this requirement is called a Cauchy problem. The imposition of Cauchy boundary conditions upon a set of partial differential equations whose characteristics are incompatible with that prescription results in an over-specification of the problem, a non-unique solution, and physically the resulting solution will not preserve any of the symmetries which were initially imposed. The classical wave equation serves as a prototype for second-order partial differential equations which accept Cauchy-type data and are called by geometers as hyperbolic equations. Generally these equations will yield a theory of observation because most physical systems encountered have both space and time represented in the differential equation set. For this reason, the slopes of the characteristic curves are called the characteristic eigenvalues and are of interest because in the hyperbolic case they represent the group velocity of wave motion. There is an immediate discrimination between space and time in the development of the theory of characteristics which appears to be contrary to the basic dictum of relativity that space and time should not be discriminated. This apparent philosophical contradiction is one imposed by the theory of measurement in which we are required to isolate a spatial point and then integrate over time in order to gather information, and until we can develop techniques to do the reverse, we shall have to reconcile the beauty of Einstein's thinking with the rank actualities of measurement.

Therefore the relativistic theory of characteristics takes a set of non-discriminating equations and seeks to discover how a violation of that symmetry at the boundary is carried through the entire space–time plane.

A particular point of interest to physicists is that it is only upon characteristic curves which energy and momentum propagate and it is only upon characteristic curves which discontinuities propagate. All waves must carry with them discontinuities as can be argued by the following. Inside the wave packet, the disturbed region has non-zero solutions to the differential equations of interest. Outside the wave packet, in the region not yet reached by the disturbance, the solutions may be absolutely zero. Thus a Taylor expansion from either side of the boundary of the disturbance must fail to connect the two regions; hence discontinuities must appear in some derivative of the physical variable being measured or being propagated. Most waves, generally treated, have discontinuities in the higher order derivatives with continuous zero and first derivatives; however, shock waves represent discontinuities in the function itself being propagated forth. Furthermore, the slightest non-linearity in the system of equations couples the various coefficients in the Taylor expansion so that ultimately, in time, a higher order discontinuity may have its effect in the spatial part of the lower orders of the observable.

The characteristic curves are purely a function of the partial differential equation set and independent of the geometry of the boundary. By a non-linear coordinate transformation we rotate the set into diagonal form. In this diagonal frame the original collection of independent variables, viz. pressure, density, energy, is contracted into some scalar function, now determined by an ordinary differential equation. The scalar function, in hydrodynamics, is known as the Riemann invariant. The constant of integration is fixed at the boundary and the Riemann invariant is conserved along the characteristic curves which emanate from the initial data. The ripples seen when a pebble is tossed into the water represent surfaces which carry information of the initial disturbance. Since most fluids are homogeneous and isotropic, these ripples generally reflect that symmetry and appear as a set of concentric circles on the water's surface. In an ideal star, they would be concentric spheres. The ripples plotted in the space–time plane are the characteristic surfaces of the fluid. Their group velocity is $u \pm c$, in which u and c are the fluid and sound velocities, indicating that ripples may be explosive or implosive as the only two possibilities. This group velocity is the characteristic eigenvalue of the curve, and may change with the physical properties of the medium. A solution to the original partial differential equation set whose independent variables are restricted to values imposed by the characteristic equation represents the Green's function to the problem. Hence the existence of a Green's function is not a foregone conclusion and requires the existence of a solution to the characteristic differential equation set.

One can imagine all of space filled by a non-linear characteristic coordinate net. Any point in space–time is then determined by the intersection of two independent characteristic curves which can be traced to their point of origin on the boundary. The chain of points along the boundary confined between the intersection of these characteristics defines the domain of dependence of the particular point upon the initial data.

Any finite region in the boundary has emanating from it families of characteristic curves whose development into space–time define a region of influence of this initial data with sharp, causative boundaries.

The partial differential equations of classical hydrodynamics lend themselves to a hyperbolic form in which case the characteristic roots and surfaces are real (Courant & Hilbert, 1937). Their solution is a Cauchy problem over a restricted domain, thus some initial data specified on a space–time surface will have a finite range of influence beyond which points exist that can have no causal connection to that data (Courant & Friedrichs, 1948). The designation of these restricted domains of causality in classical hydrodynamics must be connected to relativistic hydrodynamics in a continuous fashion. Much as elements in the proper Lorentz group can evolve continuously only to time-like representations of the orthogonal group in three dimensions, the classical Riemann (Riemann, 1876) invariants of hydrodynamics must have a relativistic extension which remains hyperbolic. A classical Mach line separates two adjacent regions which are under the influence of independent pieces of data and across which observables take on discontinuities. The names of Riemann and Mach permeate the classical literature in this subject, therefore it appears quite natural to extend their formalism to include general relativity. The Riemann invariants have application to astrophysical catastrophies and to self-gravitating relativistic pulsations (Cohen, 1969). These applications are to be discussed in another communication. Here we deal with the logical development of these invariants. We now derive the entropy conservation law in general relativity. It being a thermodynamic statement, will follow naturally from the stress energy tensor.

2. *Analysis of Dissipating Fluid Motion*

Consider the stress tensor of a relativistic fluid in units of $c = 1$.

$$T^{\mu\nu} = [E + p + \rho] u^{\mu} u^{\nu} + g^{\mu\nu} p \tag{2.1}$$

Then a scalar reminiscent of the work may be manufactured in which the external sources of energy and momentum delivered to the fluid are denoted by F^{μ},

$$g_{\mu\alpha} u^{\alpha} T^{\mu\nu}_{;\nu} = g_{\mu\alpha} u^{\alpha} F^{\mu} \tag{2.2}$$

Baryon conservation for a fluid of similar particles is written as,

$$(u^{\nu} \rho)_{;\nu} = 0 \tag{2.3}$$

in which ρ is the rest mass density. Equation (2.2), when expanded in view of (2.3), becomes

$$g_{\mu\alpha} u^{\alpha} u^{\mu} (u^{\nu}(p + \rho + E)/\rho)_{;\nu} + g_{\mu\alpha} g^{\mu\nu} u^{\alpha} p_{;\nu} = g_{\mu\alpha} u^{\alpha} F^{\mu} \tag{2.4}$$

The contravariant material velocity four vector is normalized such that,

$$g_{\mu\alpha} u^{\alpha} u^{\mu} = -1 \tag{2.5}$$

We require that

$$g_{\mu\alpha} g^{\mu\nu} = \delta^{\nu}_{\alpha} \tag{2.6}$$

and observe the chain rule of differentiation,

$$(u^{\nu}(p + \rho + E)/\rho)_{;\nu} = ((p + \rho + E)/\rho) u^{\nu}_{;\nu} + u^{\nu}((p + E)/\rho)_{;\nu} \tag{2.7}$$

Combining these results gives,

$$-\rho u^{\nu}[(p + E)/\rho]_{;\nu} + u^{\nu} p_{;\nu} = g_{\mu\alpha} u^{\alpha} F^{\mu} - (p + \rho + E) u^{\nu}_{;\nu} \tag{2.8}$$

We define Q_0 as the heat transfer rate to the fluid in a rest frame.

$$Q_0 = g_{4\alpha} u^4 F^{\alpha} \tag{2.9}$$

In the case of a dissipating fluid, Q_0 is negative. Since E, p and ρ are scalars, their covariant derivatives reduce to the ordinary form.

This reduction yields,

$$u^\alpha[E(\ln E)_{,\alpha} - (p+E)(\ln \rho)_{,\alpha}] = Q_0 \tag{2.10}$$

Equation (2.10) is independent of the metric.

By the virial theorem, p is equal to $(\Gamma - 1)E$ in which Γ is the ratio of specific heats. Equation (2.10) then becomes,

$$Dp - a_s^2 D\rho = Q_0/(\Gamma - 1) \tag{2.11}$$

in which a_s^2 is the isentropic speed of sound and D is the classical flow derivative,

$$a_s^2 = \Gamma p/\rho \tag{2.12}$$

$$D = \frac{\partial}{c\,\partial t} + \beta_i \frac{\partial}{\partial x_i}$$

It is easily seen that the isentropic law results for an adiabatic system ($Q_0 = \text{zero}$).

We now place the continuity equation (2.3) in dimensionless form,

$$u^\nu(\ln u^\nu)_{,\nu} + u^\nu(\ln \rho)_{,\nu} = -\Gamma^\nu_{\nu\mu} u^\mu \tag{2.13}$$

Although this equation is not independent of the metric, the metric is brought to the right-hand side as an inhomogeneity. In the classical limit with spherical coordinates equation (2.13) becomes

$$\partial u/\partial r + \frac{1}{\rho}\,\partial \rho/\partial t + (u/\rho)\,\partial\rho/\partial r = -(2/r)\,u_r - \cot\theta u_\theta \tag{2.14}$$

The inhomogeneity represents the areal depletion, more familiar in cases of time independent spherical symmetry when the following conservation law results as an integral of (2.14),

$$\rho u r^2 = \text{conserved} \tag{2.15}$$

We now turn to the spatial part of the equations of motion (Latin index),

$$[(p+E+\rho)\,u^j u^\beta + g^{j\beta} p]_{;\beta} = F^j \tag{2.16}$$

Using the continuity equation and separating spatial and temporal derivatives gives,

$$[(p+E)\,u^j u^k]_{;k} + \rho u^k u^j_{;j} + g^{jk} p_{;k} + [(p+E)\,u^j u^4]_{;4} + \rho u^4 u^j_{;4} = F^j \tag{2.17}$$

In the interest of clarity, we now limit ourselves to motion in one direction, $u' \neq 0$, $u^2 = u^3 = 0$, $u^4 \neq 0$. We adopt a spherical diagonal metric in order to obtain results relevant to stellar systems,

$$g^{\mu\nu} = g^\mu \,\delta^{\mu\nu}$$

with

$$g^{11} = -1, \qquad g^{22} = -r, \qquad -g^{33} = r\sin\theta, \qquad g^{44} = 1$$

A more complete derivation may be picked up at this point;† however, most of the results will be independent of the metric tensor, hence the calculation is simplified in this system. With a slight rearrangement, these restrictions leave us with the following equation:

$$[\rho + 2(p+E)]\,u^1 u^1_{;1} + (u^1 u^1 - 1)\,p_{;1} + u^1 u^1 E_{;1} \\ + u^1 u^4 (p+E)_{;4} + (p+E+\rho)\,u^4 u^1_{;4} + (p+E)\,u^1 u^4_{;4} = F^1 \quad (2.18)$$

All the forces of external fields, gravity and radiation are now absorbed in F^1. According to Thomas (1930), the effective external force is the covariant derivative of the field stress tensor with which the fluid interacts. Most of our results are independent of F^1 and hence we assign only a generic symbol to this field. Equation (2.5) implies that,

$$u^4 u^4_{;} = u^1 u^1_{;} \quad (2.19)$$

with which we derive

$$[\rho + 2(p+E)]\,(u^1 u^1_{;1}) + u^4 u^4 p_{;1} + u^1 (u^1 E_{;1}) \\ + [\rho + (1 + (u^1/u^4)^2)(p+E)]\,(u^4 u^1_{;4}) + u^1 u^4 p_{;4} + u^1 (u^4 E_{;4}) = F^1 \quad (2.20)$$

We now utilize the thermodynamic postulate for the existence of an equation of state which is amphimorphic among its variables.

$$E(p, \rho) = \text{constant} \quad (2.21)$$

implies

$$p_{;} = (\partial p/\partial E)\,E_{;} - (\partial p/\partial E)(\partial E/\partial \rho)\,\rho_{;} \quad (2.22)$$

then Equation (2.20) becomes

$$[\rho + 2(p+E)]\,u^1 u^1_{;1} + (1 + (u^4/u^1)^2 (\partial p/\partial E))\,u^1 E_{;1} \\ - (u^4 u^4/u^1)(\partial p/\partial E)(\partial E/\partial \rho)\,u^1 \rho_{;1} \\ + (\rho + (1 + (u^1/u^4)^2)(p+E))\,u^4 u^1_{;4} + u^1((\partial p/\partial E) + 1)\,u^4 E_{;4} \\ - u^1 (\partial p/\partial E)(\partial E/\partial \rho)\,u^4 \rho_{;4} = F^1 \quad (2.23)$$

† Any element of the Lorentz group may be expressed in polar form (Yamanouchi, 1970)

$$g_{\mu\nu} = g^0_{\mu\gamma}\, G^\gamma_\nu$$

in which G is a positive symmetric matrix and g^0 is a reduced matrix which contains an element of $O3$ as a subgroup, i.e.

$$g^0_{\mu\gamma} = \left(\begin{array}{c|c} O3 & O \\ \hline O & \pm 1 \end{array}\right)$$

We therefore can redefine the pressure in case of a more general metric

$$g_{\mu\nu}\, p = g^0_{\mu\gamma}\, P^\gamma_\nu$$

in which $P^\gamma_\nu = G^\gamma_\nu\, p$. Furthermore G has zero covariant derivative so that

$$P^\gamma_{\nu;\gamma} = G^\gamma_\nu\, p_{;\gamma}$$

and this substitution into equation (2.17) will then permit an extension of this derivation into general relativity.

We now divide the entire equation by ρu^1, place the derivatives in logarithmic form, and express the invariant derivative in terms of the ordinary operator plus the Christoffel rotation coefficients. We note that for a spherical metric, the only non-vanishing rotation coefficients are those with an angular index and these are not to be found in this equation; hence the covariant derivative may be replaced by the ordinary operation.

We define the operator

$$D_i = u^i \frac{\partial}{\partial x_i} \ln \tag{2.24}$$

then,

$$\begin{aligned}\left(1 + 2\frac{p+E}{\rho}\right) D_1 u^1 + \left(1 + \frac{\partial p}{\partial E}(u_4 u_4)/(u_1 u_1)\right)\frac{E}{\rho} D_1 E \\ - u_4 u_4/(u_1 u_1)\, \partial p/\partial E\, \partial E/\partial \rho\, D_1 \rho + (1 + (1 + (u^1/u^4)^2)(p+E)/\rho)\, D_4 u^1 \\ + (\partial p/\partial E + 1)\frac{E}{\rho} D_4 E - \partial p/\partial E(\partial E/\partial \rho)\, D_4 \rho = F^1/(\rho u^1)\end{aligned} \tag{2.25}$$

3. *Discussion of the Method*

The method we employ to find the Riemann compatibility relations arises out of a general technique first developed by Schwartz (1951) to analyze discontinuous initial data. This method has been reduced by Lax (1954) for sets of equations with two independent variables in which the initial data is Lipschitz continuous. Our technique is the third simplification which can be developed in a short space which follows.

We now have three partial differential equations of first order in two independent variables (space and time) and three dependent variables (u^1, ρ, E). These three equations may be brought into matrix form if the continuity and entropy equations are expressed in terms of the logarithmic operator [2.24]. The structure which results is the following:

$$T D_1 \Psi + S D_4 \Psi = \Phi \tag{3.1}$$

in which Ψ is a three-component vector representing the state of the fluid,

$$\Psi = (\rho, u^1, E) \tag{3.2}$$

and Φ is the three-component generalized force with dimension of inverse length,

$$\Phi = (-\Gamma^{\beta}_{\beta\alpha} u^{\alpha}, F^1/(\rho u^1), Q_0/(p+E)) \tag{3.3}$$

We caution that the state vector has incommensurable components (velocity, momentum, energy) and it makes no physical sense to normalize it. It is a convenient multicomponent quantity which will evolve to the Riemann invariant. The temporal matrix has components

$$\begin{aligned} &T_{11}=1, \quad T_{12}=(u^1/u^4)^2, \quad T_{13}=0 \\ &T_{21}=-\partial p/\partial E(\partial E/\partial \rho), \quad T_{22}=1+(1+(u^1/u^4)^2)(p+E)/\rho \\ &T_{23}=\frac{E}{\rho}(1+\partial p/\partial E) \\ &T_{31}=1, \quad T_{32}=0, \quad T_{33}=-E/(p+E) \end{aligned} \tag{3.4}$$

while the spatial matrix has components,

$$\begin{aligned} &S_{11}=1, \quad S_{12}=1, \quad S_{13}=0 \\ &S_{21}=-(u^4/u^1)^2(\partial p/\partial E)(\partial E/\partial \rho), \quad S_{22}=1+2\frac{p+E}{\rho} \\ &S_{23}=(E/\rho)(1+(p/E)(u^4/u^1)^2) \quad S_{31}=1, \\ &S_{32}=0, \quad S_{33}=\frac{-E}{p+E} \end{aligned} \tag{3.5}$$

All of the matrix coefficients are real and enable us to construct a left-side inverse to the temporal matrix which conjugates the entire equation (3.1) to yield the following form

$$D_4\Psi+T^{-1}SD_1\Psi=T^{-1}\Phi \tag{3.6}$$

Since S and T consist entirely of real coefficients, so will their inverses and hence $(T^{-1}S)$ will also. If the eigenvalues and eigenvectors of $T^{-1}S$ are real, then the equations (Lax, 1953) are hyperbolic and will accept Cauchy type initial data which is at least Lipshitz† continuous. Let e_k be the left-side eigenvector which precipitates the eigenvalue λ_k from the matrix product. If

$$\Lambda\equiv T^{-1}S \tag{3.7}$$

then

$$e_k\Lambda=\lambda_k e_k \tag{3.8}$$

and upon multiplying equation (3.1) on the left by the eigenvector e_k, we obtain

$$[D_4+\lambda_k D_1](e_k.\Psi)=e_kT^{-1}\Phi \tag{3.9}$$

The characteristic directional derivative may now be defined as

$$d_k=D_4+\lambda_k D_1 \tag{3.10}$$

It has a slope in the space–time plane of

$$\frac{dx^1}{dx^4}=\frac{u^1}{u^4}\lambda_k \tag{3.11}$$

† Lipshitz continuity implies that a chord drawn between any two points of the data will have a slope which is bounded.

The quantities $e_k D\Psi$ and $e_k T^{-1}\Phi$ are now scalars; we abbreviate,

$$\Sigma = e_k T^{-1}\Phi \tag{3.12}$$

The equation of the Riemann compatibility relations is now a first-order differential equation, or one form. The classical Riemann invariants are obtained from a zero source ($\Sigma = 0$). The characteristic directional derivative may now be defined by the following expression of equation (3.9),

$$e_k d_k \Psi = \Sigma \tag{3.13}$$

or in expanded form,

$$\begin{aligned} &e_k^1 d_k\rho + e_k^2 d_k u^1 + e_k^3 d_k E \\ &\quad = e_k^1(T_{11}^{-1}\phi_1 + T_{12}^{-1}\phi_2 + T_{13}^{-1}\phi_3) + e_k^2(T_{21}^{-1}\phi_1 \\ &\qquad + T_{22}^{-1}\phi_2 + T_{13}^{-1}\phi_3) + e_k^3(T_{31}^{-1}\phi_1 + T_{33}^{-1}\phi_2 + T_{33}^{-1}\phi_3) \end{aligned} \tag{3.14}$$

The eigenvectors e_k are not orthogonal; hence the coordinate system of characteristic lines generated by equation (3.11) is not an orthogonal one. The eigenvectors e_k are left-side eigenvectors as opposed to those used in quantum mechanics which suffer the operation of differentiation. These eigenvectors rotate the differential operators in the space–time plane until all of the components of the state vector have the same value.

It is a result of this analysis that this differential one form in three components can be contracted into one of two in which case it is absolute that a single integral surface exists. Contingent upon demonstration of this contraction, we conclude that the Riemann invariants exist continuously to the extreme relativistic limit where they evolve to the light cone from their classical values.

4. *The Characteristic Directions*

After a straightforward but lengthy calculation we find the following results for the eigenvalues. One value is cofluid as is the classical case, while the other two emerge as the relativistic analogues of the forward and backward characteristics,

$$\lambda_0 = 1 \tag{4.1}$$

$$\lambda_\pm = 1 + \frac{1 \mp \dfrac{1}{\beta}\sqrt{\left[\dfrac{E}{\rho}\dfrac{\partial E}{\partial p}\dfrac{p+E+\rho}{p+E}\Big/\left(\dfrac{E}{\rho} - \dfrac{E}{p+E}\dfrac{\partial E}{\partial \rho}\right)\right]}}{\gamma^2\left[\beta^2 - \dfrac{E}{\rho}\dfrac{p+E+\rho}{p+E}\dfrac{\partial E}{\partial p}\Big/\left(\dfrac{E}{\rho} - \dfrac{E}{p+E}\dfrac{\partial E}{\partial \rho}\right)\right]}$$

in which we have redefined

$$u^i = \gamma\beta^i, \qquad \gamma = 1/\sqrt{(-g_{ij}\beta^i\beta^j)} \tag{4.2}$$

in order to facilitate reduction to the results of special relativity. We notice that no spatial terms enter into this expression.

If we now assume that the eigenvalue is the hyperbolic composition of flow and sound speeds,

$$\beta\lambda = \frac{\beta \pm \beta_s}{1 \pm \beta\beta_s} \tag{4.3}$$

Then we conclude that the relativistic speed of sound is given by

$$\beta_s = \sqrt{\left[\frac{\partial p}{\partial E}\left(\frac{p+E}{p+E+\rho} - \frac{\partial E}{\partial \rho}\frac{\rho}{p+E+\rho}\right)\right]} \tag{4.4}$$

The characteristic eigenvalues become,

$$\lambda_{\pm} = 1 \pm \frac{\beta_s}{\beta}\frac{1-\beta^2}{1 \pm \beta\beta_s} \tag{4.5}$$

It is seen that both characteristics coalesce to the light cone in the ultra-relativistic limit. Just as in the classical case, supersonic flow $\beta > \beta_s$ leads to two positive definite characteristics. Thus the initial sensing of a relativistic explosion will always be followed by an echo, which represents a negative supersonic characteristic.

Two possible simplifications of this formula may be made. In the case of an adiabatic system, the first law of thermodynamics gives the relation

$$p = -\rho\, \partial E/\partial \rho \tag{4.6}$$

In which case

$$\beta_s = \sqrt{\left[\left(\frac{\partial p}{\partial E}\right)_s \frac{E}{p+E+\rho}\right]} \qquad \text{for } dQ = 0 \tag{4.7}$$

Otherwise the formula may be contracted in terms of the average kinetic energy per particle.

$$\rho\frac{\partial}{\partial \rho}(E/\rho) = \rho\, \partial E/\partial \rho - E \tag{4.8}$$

and if

$$E/\rho = \epsilon \tag{4.9}$$

then

$$\beta_s = \sqrt{\left(\frac{\partial p}{\partial E}\frac{p - \partial\epsilon/\partial\rho}{p+E+\rho}\right)} \tag{4.10}$$

The energy ϵ is the internal energy per particle from which the rest mass energy has been removed. The speed of sound always remains real as long as $\partial p/\partial E$ is positive. Its reality guarantees a causal relation between any space–time point and some initial data on the boundary. There seems to be no evidence that this derivative can go negative, in which case the characteristics would be imaginary and no information would propagate out of such a system. The original calculations of Chandrasekhar (1957) of the relativistic statistical mechanics of an electron gas permit no such possibility, although I should remark that these calculations were based upon the assumed decoupling of the Dirac equation into large and small components, which loses its validity in the ultrarelativistic limit. It therefore might remain to be investigated if the statistical integrals in this limiting case could reverse the sign of the energy in which case the system would be acausal, being propagated into the past providing the possibility for the creation of antiparticles.

We now examine various kinetic models. The Maxwell–Juttner model of a non-degenerate ideal gas is reviewed in Chandrasekhar (1957). It has three interesting limits; hence three limiting equations of state, derived from the virial theorem.

(A)	$E = \frac{3}{2}p\left(1 + \frac{5}{4}\left(\frac{p}{\rho}\right)^2\right)$	warm relativistic monatomic gas ($kT \ll m$)
(B)	$E = p/(\Gamma - 1)$	cold relativistic gas (Γ = ratio of specific heats)
(C)	$E = p/3$	hot relativistic gas ($\Gamma = 4/3$ in B) (4.11)

Reductions B and C require no dependence of the internal energy upon density, the speed of sound then reduces to

$$\beta_s \to \surd[(\Gamma - 1)\Gamma p/(\Gamma p + (\Gamma - 1)\rho)] \tag{4.12}$$

which in the individual cases reduces further to

$$\text{(B)} \quad \rho \gg p + E, \ \surd(\Gamma p/\rho) \leftarrow \beta_s \to \surd(\Gamma - 1), \ \rho \ll p + E \quad \text{(C)} \tag{4.13}$$

These are not extreme limits, since the intermediate case of the Maxwell–Juttner gas gives the following first-order correction for finite density dependence

$$\beta_s = \beta_0 \surd\left(\frac{1 + (27/4)\beta_0^4}{(1 + (45/4)\beta_0^4)(1 + (3/2)\beta_0^2 + (27/8)\beta_0^6)}\right) \tag{4.14}$$

$$\beta_0^2 = \gamma p/\rho$$

valid for systems in which the thermal energy is comparable to the rest energy, i.e.,

$$kT \sim m_e c^2 \sim 6 \times 10^9 \text{ °K} \tag{4.15}$$

for an electron gas. For systems with temperatures in excess of a billion degrees, the formula given in Chandrasekhar (1957) may be used and the speed of sound will be a complicated function of those integrals which are expressible as combinations of Hankel functions. In the Maxwell–Juttner case, the energy-density change is negative, and we caution that this derivative is taken at constant pressure, not at constant entropy. We have avoided the use of isentropic derivatives in order to include inhomogeneities of dissipation, external forces and geometric curvature.

Degenerate gases will require appropriate equations of state with which to evaluate the terms in the speed of sound. In this case the result is a complicated function of the Fermi integrals (Chandrasekhar, 1957).

5. *The Riemann Compatibility Relations*

The left-side eigenvector which will diagonalize the fluid matrix along non-redundant directions, has the following components

$$\begin{aligned} e_{\pm}^{(0)} &= \frac{E}{p+E}\frac{\partial p}{\partial E} \\ e_{\pm}^{(1)} &= -\frac{\partial E}{\partial \rho} e_{\pm}^{(0)} \\ e_{\pm}^{(2)} &= \pm\left(\frac{E}{\rho} + \frac{E}{p+E}\right)\beta\beta_s \\ e_{\pm}^{(3)} &= \frac{E}{\rho} e_{\pm}^{(0)} \end{aligned} \tag{5.1}$$

in which $e_{\pm}^{(0)}$ is a convenience. We note that only $e_{\pm}^{(2)}$ actually is different for forward and backward characteristics. The proportionality factor between vectors one and three is an integrating factor which reduces a classical Pfaffian of three dimensions to one of two (Forsyth, 1959). This reduction fulfills necessary and sufficient conditions for the existence of a single integral surface upon which these invariants will lie for all relativistic conditions. Thus they exist in all cases. Equation (3.14) may now be written

$$-e_{\pm}^{(0)}\frac{\partial E}{\partial \rho} d\ln\rho + e_{\pm}^{(0)}\frac{E}{\rho} d\ln E + e_{\pm}^{(2)} d\ln u^1 = \Sigma \tag{5.2}$$

If we carry out the reduction implied by equation (3.14) and expand the results, we obtain the following differential equation for the inhomogeneous Riemann compatibility relations which are propagated along characteristics

$$\begin{aligned} &\frac{p}{\rho} d\ln p \pm \left(\frac{p+E}{\rho}+1\right)\beta_s \frac{d\beta}{1-\beta^2} \\ &= \frac{\beta \pm \beta_s}{1-\beta_s^2}\left[\pm\beta\beta_s\frac{F/\rho}{\gamma\beta} - \beta_s^2\left(1+\frac{p+E}{\rho}(1\pm\beta/\beta_s)\right)\frac{2}{r}\gamma\beta - \left(\frac{\partial p}{\partial E}\pm\beta\beta_s\right)Q_0/\rho\right] \end{aligned} \tag{5.3}$$

in which we have dropped the superscript in the external force. It is understood to be a vector colinear with β and its form may be that suggested by Oppenheimer & Volkoff (1939).

The derivative is taken along a characteristic direction, and in integrating the radial dependence of the source we must utilize the relation

$$d = \frac{\partial}{c\,\partial t} + \frac{\beta \pm \beta_s}{1 \pm \beta\beta_s}\frac{\partial}{\partial r} \tag{5.4}$$

For a cold gas, moving at relativistic velocity we require $\rho \gg p + E$, in which case the left-hand side of equation (5.3) becomes

$$\beta_s^0 \, d\ln p/\Gamma \pm \frac{d\beta}{1-\beta^2} \tag{5.5}$$

in which

$$\beta_s^0 = \sqrt{(\Gamma p/\rho)} \tag{5.6}$$

If the system is also isentropic, then

$$\beta_s^0 \sim p^{(\Gamma-1)/\Gamma} \tag{5.7}$$

and the following relation results,

$$\frac{\beta_s^0}{\Gamma - 1} \pm \tfrac{1}{2} \ln\left(\frac{1+\beta}{1-\beta}\right) \qquad \text{is conserved} \tag{5.8}$$

For a slowly moving gas (small β) the classical result (Courant & Friedrichs, 1948) is obtained

$$\frac{\beta_s^0}{\Gamma - 1} \pm \beta \qquad \text{is conserved} \tag{5.9}$$

For a hot relativistic gas, $p + E \gg \rho$, the left-hand side of the differential equation becomes

$$\frac{dp}{p+E} \pm B_s \frac{d\beta}{1-\beta^2} \tag{5.10}$$

in which the speed of sound becomes

$$B_s = \sqrt{\left[\frac{\partial p}{\partial E}\left(1 - \frac{\partial E}{\partial p}\right)\right]} \to \sqrt{[\Gamma - 1]} \to 1/\sqrt{3} \tag{5.11}$$

Employing the virial theorem produces the following invariant in the absence of inhomogeneities

$$\frac{1+\beta}{1-\beta} p^{\pm k} \tag{5.12}$$

in which

$$k = 2(\Gamma - 1)/(\Gamma B_s) \to \sqrt{(3)}/2 \tag{5.13}$$

The inhomogeneity in (5.3) contains a geometric term which is dilated at high velocity while the effect of the external force is contracted, thus at high velocity the motion is dominated by pure geometry and not gravity. Assuming no explicit time dependence occurs in the fluid parameters, equation (5.3) reduces to the following

$$\frac{p}{\rho} d\ln p \pm \left(\frac{p+E}{\rho} + 1\right)\beta_s \frac{d\beta}{1-\beta^2} = -\beta_s^2 \frac{\beta \pm \beta_s}{1-\beta_s^2}\left[1 + \frac{p+E}{\rho}(1 \pm \beta/\beta_s)\right]\frac{2\beta}{\sqrt{(1-\beta^2)}}\frac{dr}{r} \tag{5.14}$$

Information emitted in the form of compression waves will travel along positive or negative characteristics depending upon whether or not the source approaches or recedes from us. In this case the appropriate plus or minus sign in equation (5.14) must be used. In all probability the negative characteristic curves may be more of interest to cosmic ray physicists who build models and to cosmologists concerned with primordial events.

Consider a hot plasma, then equation (5.14) reduces to the following along a negative characteristic,

$$\frac{p}{p+E}d\ln p - \beta_s \frac{d\beta}{1-\beta^2} = -\beta_s^2 \frac{\beta-\beta_s}{1-\beta_s^2}(1-\beta/\beta_s)\frac{2\beta}{\sqrt{(1-\beta^2)}}\frac{dr}{r} \quad (5.15)$$

This is a differential one form of three components whose exterior derivative does not vanish. Therefore a solution exists only as the intersection of two integral surfaces. According to the method of Euler (see Forsyth, 1959) we consider a part of the one form which has zero exterior derivative, and set $dp = 0$. This results in the following differential equation,

$$c_1(1-\beta_s^2)\frac{d\beta}{\sqrt{(1-\beta^2)}}\left[\frac{1}{\beta_s^2\beta} - \frac{1}{\beta_s^2(\beta-\beta_s)} - \frac{1}{\beta_s(\beta-\beta_s)^2}\right] = \frac{2dr}{r} \quad (5.16)$$

in which c_1 is a separation constant. If we examine the limit $\beta < \beta_s$, only the first term in the bracket remains which integrates to

$$\beta = \frac{2}{r^k + r^{-k}} \quad (5.17)$$

in which

$$k = \beta_s^2/(1-\beta_s^2) < 1/2 \quad (5.18)$$

and $\partial\beta/\partial r < 0$, i.e. there is a deceleration with distance. The relativistic horizon is given by $r = 1$, and the relation gives the velocity β, as a monatomic function of the distance.

This distance velocity relation for ultrasonic hot matter flowing on negative characteristics becomes the following

$$\beta = 2r^k/(r^{2k}+1) \simeq r^k < \sqrt{(r)}, \qquad r < 1 \quad (5.19)$$

If we now seek the ultrasupersonic limit $\beta_s < \beta \sim 1$, the equation for the Riemann invariant reduces to

$$\frac{d\beta}{\sqrt{(1-\beta^2)}} = \frac{2\beta_s^2}{1-\beta_s^2}\frac{dr}{r} \quad (5.20)$$

which accepts the integral

$$\beta = \cos\left(\frac{\beta_s^2}{1-\beta_s}\ln\frac{1}{r^2}\right) \quad (5.21)$$

This is an oscillatory terminal phase in which the velocity increases with distance while decelerating.

The complete integration of equation (5.16) yields the following integral surface

$$\frac{\beta_s}{1-\beta_s^2}(\operatorname{sech}^{-1}\beta_s)+\frac{\log 2-\log((\beta-\beta_s)/(1-\beta\beta_s))}{\sqrt{(1-\beta_s^2)\beta_s^2}}+c_1\ln r^2=c_2 \quad (5.22)$$

Then the remaining part of the differential one form (equation (5.15)) may be integrated into a form of equation (5.8).

$$\frac{\beta_s^0}{\Gamma-1}\pm(1-c_1)\tfrac{1}{2}\ln\frac{1+\beta}{1-\beta}=c_3 \quad (5.23)$$

assuming the virial theorem relates pressure to energy. These are two integral surfaces related by the separation constant c_1, which can be removed to yield

$$\frac{\beta_s}{1-\beta_s^2}(\operatorname{sech}^{-1}\beta)+\frac{\log 2-\log((\beta-\beta_s)/(1-\beta\beta_s))}{\sqrt{(1-\beta_s^2)\beta_s^2}}$$
$$\pm\left\{\frac{c_3-\beta_s^0/(\Gamma-1)}{\frac{1}{2}\ln((1+\beta)/(1-\beta))}\mp 1\right\}\times\ln\frac{2}{r}=c_2 \quad (5.24)$$

The constants c_2 and c_3 are integration constants to be fixed at a boundary.

What is disguised in these integrals is the acceleration process which occurs on the negative characteristic. According to equation (5.12), we have the following homogeneous Riemann invariant

$$\frac{1-\beta_s}{1+\beta_s}p^{-k}=\frac{1-\beta_s^{(0)}}{1+\beta_s^{(0)}}p_0^{-k} \quad (5.25)$$

in which the index zero stands for an initial value. If the fluid decelerates as it leaves the original event, as it must from a thermodynamic argument, then we have that

$$\beta_s^{(0)}>\beta_s \quad (5.26)$$

It then follows that, $p>p_0$.

In other words, if the fluid decelerates of its own accord, then the kinetic energy of organized flow is fed into internal energy and the fluid gets hotter. This effect is independent of the geometrical inhomogeneity introduced by the Christoffel rotation term. However, even considering that inhomogeneity (equation (5.24)) we see that there will be a further enhancement of the energy as the fluid approaches the observer while slowing down. In other words, the structure of the inhomogeneous Riemann invariant upon the negative characteristic is of the form,

$$d\left(\frac{1-\beta}{1+\beta}\frac{1}{p^k}\right)=K(r) \quad (5.27)$$

Hence, as K increases, while β decreases, the relativistic pressure and internal energy must increase.

Although this is an apparent physical interpretation of these equations, their full physical meaning remains to be explored.

Acknowledgements

The author is indebted to Dr. A. H. Taub and Dr. B. F. Schutz, Jr., whose review of this manuscript clarified many points, and brought to the surface issues of controversy.

References

Allen, W. (1954). *Astrophysical Quantities*, Chapter 14, p. 242. Athlone Press, London.

Chandrasekhar, S. (1957). *An Introduction to the Study of Stellar Structure*, Chapter 10, p. 394. Dover Pub. Co., New York.

Cohen, J. M. (1969). *International Journal of Theoretical Physics*, Vol. 3, No. 4, p. 267.

Courant, R. and Friedrichs, K. O. (1948). *Supersonic Flow and Shock Waves*, Chapter II, p. 48, and Chapter III, p. 87. Interscience, New York.

Courant, R. and Hilbert, D. (1937). *Methods of Mathematical Physics*, German edition, Vol. II, Chapter 50, p. 313. Springer-Verlag, Berlin.

Flanders, H. (1963). *Differential Forms with Applications to the Physical Sciences*, Chapter I, p. 2. Academic Press, N.Y.

Forsyth, A. W. (1959). *Theory of Differential Equations*, Vol. I, Chapter I, p. 11. Dover Pub. Co., New York.

Lax, P. D. (1953). *Communications on Pure and Applied Mathematics*, **6**, 231.

Lax, P. D. (1954). *Annals of Mathematics Studies*, **33**, 211.

Oppenheimer, J. R. and Volkoff, G. M. (1939). *Physical Review*, **55**, 374.

Riemann, G. F. B. (1876). *Gesammelte Werke*, Chapter VIII, p. 145, and Chapter IX. Teubner-Verlag, Leipzig.

Schwartz, L. (1951). *Theorie des Distributions*, Actualities Scientifiques et Industrielles, Vol. II, No. 1122, p. 137 (see also Vol. I, No. 1245, 1957); *Mathematics for the Physical Sciences* (1966). Paris-Hermann.

Taub, A. H. (1948). *Physical Review*, **74**, 328.

Thomas, L. H. (1930). *Quarterly Journal of Mathematics*, **1**, 239.

Yamanouchi, T. (1970). *Mathematics Applied to Physics* (edited by G. A. Deschamps *et al.*), LC-78-79553, Chapter X, p. 595. Springer-Verlag, New York.